

8. Discrete time processing of continuous time signals

Even though this course is primarily about the discrete time signal processing, most signals we encounter in daily life are continuous in time such as speech, music and images. Increasingly discrete-time signals processing algorithms are being used to process such signals. For processing by digital systems, the discrete time signals are represented in digital form with each discrete time sample as binary word. Therefore we need the analog to digital and digital to analog interface circuits to convert the continuous time signals into discrete time digital form and vice versa. As a result it is necessary to develop the relations between continuous time and discrete time representations.

1. Sampling of continuous time signals:

Let $\{x_c(t)\}$ be a continuous time signal that is sampled uniformly at $t = nT$ generating the sequence $\{x[n]\}$ where

$$x[n] = x_c(nT), \quad -\infty < n < \infty, \quad T > 0$$

T is called sampling period, the reciprocal of T is called the sampling frequency $f_s = 1/T$. The frequency domain representation of $\{x_c(t)\}$ is given by its Fourier transform

$$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t)e^{-j\Omega t} dt$$

where the frequency-domain representation of $\{x[n]\}$ is given by its discrete time fourier transform

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$$

To establish relationship between the two representation, we use impulse train sampling. This should be understood as mathematically convenient method for understanding sampling. Actual circuits can not produce continuous time impulses. A periodic impulse train is given by

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{8.1}$$

FIGURE 8.1

$$x_p(t) = x_c(t)p(t) \quad (8.2)$$

using sampling property of the impulse $f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$, we get

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) \quad (8.3)$$

FIGURE 8.2

From multiplication property, we know that

$$X_p(j\Omega) = \frac{1}{2\pi} [X_c(j\Omega) \star P(j\Omega)]$$

The Fourier transform of a impulse train is given by

$$P(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

where $\Omega_s = \frac{2\pi}{T}$

Using the property that $X(j\Omega) \star \delta(\Omega - \Omega_0) = X(j(\Omega - \Omega_0))$ it follows that

$$X_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j\Omega - k\Omega_s) \quad (8.4)$$

Thus $X_p(j\Omega)$ is a periodic function of Ω with period Ω_s , consisting of superposition of shifted replicas of $X_c(j\Omega)$ scaled by $1/T$. Figure 8.3 illustrates this for two cases.

FIGURE 8.3

If $\Omega_m < (\Omega_s - \Omega_m)$ or equivalently $\Omega_s > 2\Omega_m$ there is no overlap between shifted replicas of $X_c(j\Omega)$, where as with $\Omega_s < 2\Omega_m$, there is overlap. Thus if $\Omega_s > 2\Omega_m$, $X_c(j\Omega)$ is faithfully replicated in $X_p(j\Omega)$ and can be recovered from $x_p(t)$ by means of lowpass filtering with gain T and cut off frequency between Ω_m and $\Omega_s - \Omega_m$. This result is known as Nyquist sampling theorem.

Sampling Theorem: Let $x_c(t)$ be a bandlimited signal with $X_c(j\Omega) = 0$, for $|\Omega| > \Omega_m$. Then $X_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT)$, $-\infty < n < \infty$, if

$$\Omega_s = \frac{2\pi}{T} > 2\Omega_m$$

The frequency $2\Omega_m$ is called Nyquist rate, while the frequency Ω_m is called the Nyquist frequency.

The signal $x_c(t)$ can be reconstructed by passing $x_p(t)$ through a lowpass filter.

FIGURE 8.4

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \leq \Omega_c \\ 0, & |\Omega| \geq \Omega_c \end{cases}$$

The impulse response of this filter is

$$\begin{aligned} h_r(t) &= \frac{\sin \Omega_c t}{\pi t/T} \\ x_r(t) &= x_p(t) \star h_r(t) \\ &= \left(\sum_{n=-\infty}^{\infty} x_c(nT) \delta(t - nT) \right) * h_r(t) \\ &= \sum_{n=-\infty}^{\infty} x_c(nT) h_r(t - nT) \end{aligned} \tag{8.5}$$

Assuming $\Omega_c = \Omega_s/2 = \pi/T$ we get

$$x_r(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T} \tag{8.6}$$

The above expression (8.5) shows that reconstructed continuous time signal $x_r(t)$ is obtained by shifting in time the impulse response of low pass filter $h_r(t)$ by an amount nT and scaling it in amplitude by a factor $x[n] = x_c(nT)$ for all integer values n . The interpolation using the impulse response of an ideal low pass filter in (8.6) is referred to as bandlimited interpolation, since it implements reconstruction if $x_c(t)$ is bandlimited and sampling frequency satisfies the condition of the sampling theorem. The reconstruction is in the mean square sense i.e.

$$\int_{-\infty}^{\infty} (x_c(t) - x_r(t))^2 dt = 0$$

2. The effect of undersampling: Aliasing

We have seen earlier that spectrum $X_c(j\Omega)$ is not faithfully copied when $\Omega_s < 2\Omega_m$. The terms in (8.4) overlap. The signal $x_c(t)$ is no longer recoverable

from $x_p(t)$. This effect, in which individual terms in equation (8.4) overlap is called aliasing.

For the ideal low pass signal

$$h_r(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Hence $x_r(nT) = x_c(nT)$, $n = 0, \pm 1, \pm 2, \dots$

Thus at the sampling instants the signal values of the original and reconstructed signal are same for any sampling frequency.

3. DTFT of the discrete time signal:

Taking continuous time Fourier transform of equation (8.3) we get

$$X_p(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega nT} \quad (8.7)$$

Since $x[n] = x_c(nT)$, we get the DTFT

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} \quad (8.8)$$

comparing them we see that

$$X_p(j\Omega) = X(e^{jw})|_{w=\Omega T} = X(e^{j\Omega T})$$

using equation (8.4) we get

$$X(e^{j\Omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

or

$$X(e^{jw}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{w}{T} - \frac{2\pi k}{T}\right)\right) \quad (8.9)$$

Comparing equation (8.4) and (8.9) we see that $X(e^{jw})$ is simply a frequency scaled version of $X_p(j\Omega)$ with frequency scaling specified by $w = \Omega T$. This can be thought of as a normalization of frequency axis so that frequency $\Omega = \Omega_s$ in $X_p(j\Omega)$ is normalized to $w = 2\pi$ in $X(e^{jw})$. For the example in

figure 8.3 the $X(e^{jw})$ is shown in figure (8.5)
 From equation (8.5) we see that

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]h_r(j\Omega)e^{-j\Omega T n}$$

FIGURE 8.5

$$= H_r(j\Omega)X(e^{j\Omega T}) \quad (8.10)$$

We refer to the system that implements $x[n] = x_c(nT)$ as ideal continuous-to-discrete time (C/D) convertor and is depicted in figure (8.6)

FIGURE 8.6

The ideal system that takes $\{x[n]\}$ sequence as input and produces $x_r(t)$ given equation (8.5) is called ideal discrete to continuous time (D/C) convertor and is depicted in Figure (8.7)

FIGURE 8.7

4. Discrete time processing of continuous time signal

Figure (8.8) shows a system for discrete time processing of continuous time system

FIGURE 8.8

The over all system has $x_c(t)$ as input and $y_r(t)$ as output. We have the following relations among the signals.

$$\begin{aligned} x[n] &= x_c(nT) \\ X(e^{jw}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{w}{T} - \frac{2\pi k}{T}\right)\right) \\ y_r(t) &= \sum_{n=-\infty}^{\infty} y[n] \sin \frac{\pi(t - nT)/T}{\pi(t - nT)/T} \end{aligned}$$

and

$$\begin{aligned} Y_r(j\Omega) &= H_r(j\Omega)Y(e^{j\Omega T}) \\ &= \begin{cases} TY(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

If the discrete time system is LTI then we have

$$Y(e^{jw}) = H(e^{jw})X(e^{jw})$$

combining these equations we get

$$\begin{aligned} Y_r(j\Omega) &= H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T}) \\ &= H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right) \end{aligned} \quad (8.10)$$

If $X_c(j\Omega) = 0$, for $|\Omega| \geq \pi/T$ and we use ideal lowpass reconstruction filter then only the term for $k = 0$ is passed by the filter and we get

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$

Thus if $X_c(j\Omega)$ is bandlimited and sampling rate is above the Nyquist rate, the output is related to the input by

$$Y_r(j\Omega) = H_{eq}(j\Omega)X_c(j\Omega)$$

where

$$H_{eq}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases} \quad (8.11)$$

That is overall system is equivalent to a linear time invariant system for bandlimited signal.

The LTI property of the system depends on two factors. First the discrete time system is LTI and second the input signals are bandlimited to half the sampling frequency

Example: Let us consider the system in figure 8.8 with

$$H(e^{jw}) = \begin{cases} 1 & |w| < w_c \\ 0 & w_c < |w| \leq \pi \end{cases}$$

The frequency response is periodic with period 2π . For a bandlimited input signal, sampled above the Nyquist rate, the overall system will behave like a LTI continuous time system with

$$H_{eq}(j\Omega) = \begin{cases} 1 & |\Omega T| < w_c \text{ or } |\Omega| < w_c/T \\ 0 & |\Omega| \geq w_c/T \end{cases}$$

Thus the equivalent system is ideal lowpass system with cut off frequency w_c/T . With a fixed discrete time filter by changing T we can change the cut off frequency of the equivalent system. Spectra for various signals are depicted in figure 8.9.

FIGURE 8.9

From figure (8.9) we can see that even if there is same aliasing due to sampling, if the components are filtered out by the discrete time system, the overall transfer function will remain same. Thus the requirement is

$$(2\pi - \Omega_m T) > w_c$$

instead of $(2\pi - \Omega_m T) > \Omega_m T$ for no aliasing.

5. Continuous time processing of discrete time signals:

Consider the system shown in figure (8.10)

Figure 8.10

We have

$$\begin{aligned} X_c(j\Omega) &= TX(e^{j\Omega T}), & |\Omega| < \pi/T \\ Y_c(j\Omega) &= H_c(j\Omega)X_c(j\Omega), & |\Omega| < \pi/T \\ Y(e^{jw}) &= \frac{1}{T}Y_c(j\frac{w}{T}), & |w| < \pi \\ &= H_c(j\frac{w}{T})X(e^{jw}), \end{aligned}$$

Therefore the overall system behaves as a discrete time system where frequency response is

$$H(e^{jw}) = H_c(j\frac{w}{T}), \quad |w| < \pi \quad (8.12)$$

Example: Let us consider a discrete time system with frequency response

$$H(e^{jw}) = e^{-jw\Delta}, \quad |w| < \pi$$

when Δ is an integer, this system is delay by Δ

$$y[n] = x[n - \Delta]$$

but when Δ is not an integer, we can not write the above equation. Suppose that we implement this using system in figure (8.10). Then we have

$$H_c(j\Omega) = H(e^{j\Omega T}) = e^{-j\Omega\Delta T} \quad (8.13)$$

So that overall system has frequency response $e^{-j\omega\Delta}$. The equation (8.13) represents a time delay ΔT secs in continuous time whether Δ is integer or not, thus

$$Y_c(t) = x_c(t - \Delta T)$$

The signal $x_c(t)$ is bandlimited interpolation of $x[n]$ and $y[n]$ is obtained by sampling $y_c(t)$. Thus $y[n]$ are samples of band limited signal $x_c(t)$ delayed by ΔT .

$$\begin{aligned} y[n] = y_c(nT) &= x_c(nT - \Delta T) \\ &= \sum_{k=-\infty}^{\infty} x[k] \cdot \frac{\sin[\pi(t - \Delta T - kT)/T]}{\pi(t - \Delta T - kT)/T} \Big|_{t=nT} \\ &= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n - k - \Delta)]}{\pi(n - k - \Delta)} \end{aligned}$$

For $\Delta = 1/2$, $\{y[n]\}$ are depicted in figure (8.11)
FIGURE 8.11

6. Sampling of discrete time Signals:

In analogy with continuous time sampling, the sampling of a discrete time signal can be represented as shown in figure 8.12

FIGURE 8.12

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

$$x_p[n] = \begin{cases} x[n], & \text{if } n \text{ is integer multiple of } N \\ 0, & \text{otherwise} \end{cases} \quad (8.14)$$

$$\begin{aligned} x_p[n] &= x[n]p[n] \\ &= \sum_{k=-\infty}^{\infty} x[kN]\delta[n - kN] \end{aligned}$$

In frequency domain, we get

$$X_p(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{j\theta}) X(e^{j(\omega-\theta)}) d\theta$$

The Fourier transform of $\{p[n]\}$ sequence is

$$P(e^{jw}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(w - kw_s),$$

where $w_s = \frac{2\pi}{N}$. Thus we get

$$X_p(e^{jw}) = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j(w-kw_s)}) \quad (8.15)$$

Figure 8.13 illustrates signals and their spectra
FIGURE 8.13

If $(w_s - w_m) > w_m$ or equivalently $w_s > 2w_m$ or $\frac{2\pi}{N} > 2w_m$ there will be no aliasing (i.e non zero portions of $X(e^{jw})$ do not overlap) and the signal $\{x[n]\}$ can be recovered from $x_p[n]$ by passing through an ideal low-pass filter with gain equal to N and cut off equal to $w_s/2$

$$H_r(e^{jw}) = \begin{cases} N & |w| < w_c \\ 0, & w_c < |w| \leq \pi \end{cases}$$

FIGURE 8.14

If $w_s < 2w_m$, there will be aliasing, and so $\{x_r[n]\}$ will be different from $\{x[n]\}$. However as in continuous time case

$$x_r[kN] = x[kN], \quad k = 0, \pm 1, \pm 2, \dots$$

independently of whether there is aliasing or not.

$$\begin{aligned} x_r[n] &= \{x_p[n]\} \star \{h_r[n]\} \\ &= \left\{ \sum_{k=-\infty}^{\infty} x[kN] \delta[n - kN] \right\} \star \{h_r[n]\} \\ &= \sum_{k=-\infty}^{\infty} x[kN] h_r[n - kN] \end{aligned}$$

For ideal low pass filter

$$h_r[n] = \frac{Nw_c}{\pi} \frac{\sin w_c n}{w_c n}$$

with $w_c = \pi/N$ we get

$$x_r[n] = \sum_{k=-\infty}^{\infty} x[kN] \frac{\sin \pi(n - kN)/N}{\pi(n - kN)/N}$$

7. Discrete time decimation and interpolation:

The sampled signal in equation (8.13) has $(N - 1)$ samples out of every N samples as zeros. We define a new sequence which retains only the non zero values

$$\begin{aligned} x_d[n] &= x_p[nN] \\ &= x[nN] \end{aligned} \quad (8.16)$$

this is called a decimated sequence, whatever may be the value of $N \geq 2$. The DTFT of the decimated request is given by

$$\begin{aligned} X_d(e^{jw}) &= \sum_{n=-\infty}^{\infty} x_d[n]e^{-jwn} \\ &= \sum_{n=-\infty}^{\infty} x_p[nN]e^{-jwn} \end{aligned}$$

since only for multiples of N , $x_p[n]$ has non zero value,

$$\begin{aligned} &= \sum_{m=-\infty}^{\infty} x_p[m]e^{-j\frac{w}{N}m} \\ &= X_p(e^{j\frac{w}{N}}) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j(\frac{w}{N} - \frac{2\pi k}{N})}) \end{aligned} \quad (8.17)$$

For the signal shown in figure (8.13) the sequence $\{x_d[n]\}$ and its spectrum are illustrated in figure (8.15)

FIGURE 8.15

If the original signal $\{x[n]\}$ was obtained by sampling a continuous time signal, the process of decimation can be viewed as reduction in the sampling rate by a factor of N . With this interpretation, the process of decimation is often referred as down sampling. The block diagram for this is shown in figure (8.16)

FIGURE 8.16

There are situations in which it is useful to convert a sequence to a higher equivalent sampling rate. This process is referred to as upsampling or interpolation. This process is reverse of the downsampling. Given a sequence

$\{x[n]\}$ we obtain an expanded sequence $x_e[n]$ by inserting $(L - 1)$ zero.

$$x_e[n] = \begin{cases} x[\frac{n}{L}], & n \text{ multiple of } L \\ 0, & \text{otherwise} \end{cases} \quad (8.18)$$

The interpolated sequence $\{x_i[n]\}$ is obtained by low pass filtering of $\{x_e[n]\}$

$$\begin{aligned} X_e(e^{jw}) &= \sum_{n=-\infty}^{\infty} x_e[n] e^{-jwn} \\ &= \sum_{m=-\infty}^{\infty} x[m] e^{-jwLm} \\ &= X(e^{jwL}) \end{aligned}$$

After low pass filtering

$$X_i(e^{jw}) = H(e^{jw}) X(e^{jwL}) \quad (8.19)$$

For ideal low-pass filter with cutoff $\frac{\pi}{L}$ and gain L we get

$$X_i(e^{jw}) = \begin{cases} L X(e^{jwL}), & |w| < \pi/L \\ 0, & \pi/L \leq |w| \leq \pi \end{cases} \quad (8.20)$$

Signals and their spectra interpolation are shown in figure (8.17)

FIGURE 8.17

We can get a non integer change in rate if it is ratio of two integers by using upsampling and downsampling operations.